Final Exam

1. Define and/or fully explain the following terms to prove your understanding [30 pts: 3 pts each]:
   A. commutator
   B. cofactor
   C. modal matrix
   D. unitary matrix
   E. linear operators
   F. homogeneous function of degree p
   G. Fick's law of diffusion
   H. solid angle
   I. norm
   J. basis set of a vector space

2. Let $f(\hat{A})$ be a polynomial function of $\hat{A}$: $f(\hat{A}) = a_0 + a_1\hat{A} + a_2\hat{A}^2 + \cdots + a_N\hat{A}^N$. Show that if ψ is an eigenfunction of $\hat{A}$ with eigenvalue of β, then $f(\hat{A})\psi = f(\beta)\psi$. [8 pts]

3. A Hermitian operator, $\hat{A}$, satisfies that $\int f^*(x)\hat{A}g(x)dx = \int g(x)[\hat{A}f(x)]^*dx$. Derive this relation from the following equality: $\int f^*(x)\hat{A}f(x)dx = \int f(x)[\hat{A}f(x)]^*dx$. [12 pts]

4. i) Find the eigen energy of a particle in a 3-dimensional box that is described by the wave function, $\Psi(x,y,z) = \left(\frac{4}{a^3}\right)^{1/2} sin\frac{n_x\pi x}{2a} sin\frac{n_y\pi y}{2a} sin\frac{n_z\pi z}{a}$. [5 pts] ii) Describe the lowest 5 energy levels in an energy diagram with the quantum numbers and degeneracy specified. [5 pts]

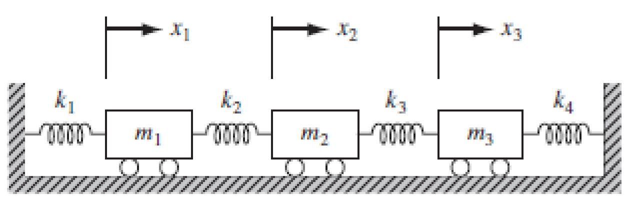
5. The electrostatic potential (φ) produced by a dipole moment ($\vec{\mu}$) located at the origin is given as $\varphi(\vec{r}) = \varphi(x,y,z) = (\vec{\mu}\cdot\vec{r})\, r^{-3}$. Derive a simple expression for the electric field ($\vec{E}$) associated with this potential. [10 pts]

6. i) For a rectangular membrane obeying the following wave equation, obtain the general solution, u=u(x,y,t) that is subject to the boundary conditions: u(0,y)=u(a,y)=u(x,o)=u(x,b)=0 for all t. [15 pts]

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

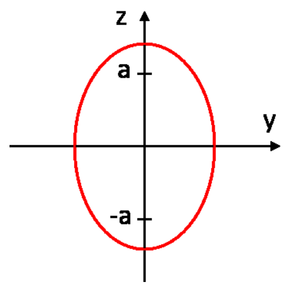
   ii) Determine and briefly discuss the vibrational behaviour of the membrane if it is initially displaced according to $u(x,y,0) = \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$ and then released from rest. [5 pts]

7. Consider the three-mass, four-spring system shown in the figure to the right: $m_1 = 8$, $m_2 = 2$, $m_3 = 2$; $k_1 = 6$, $k_2 = 2$, $k_3 = k_4 = 1$. i) Set up Newton's equations of motion by finding the potential energy stored in the system. [5 pts] ii) Determine the fundamental frequencies associated with the system. [5 pts] iii) Determine the equations for the three normal



modes. [5 pts] iv) Explain the vibrational pattern of each normal mode by sketching the displacements of the masses. [5 pts] (Assume that each spring obeys Hooke's law; Ignore any frictional forces; x_j denotes the displacement of m_j from its equilibrium position)

8. As the spherical coordinates, $\vec{r}(\mathrm{r}, \theta, \varphi)$, suit well the hydrogen atom system, a prolate spheroidal coordinate system, $\vec{r}(\eta, \theta, \varphi)$, can be useful in describing a molecular hydrogen ion (H_2^+). A prolate spheroid is obtained by rotating the ellipse in the figure to the right about its long axis that is aligned along the z-axis. The distance between the two foci on the z-axis is 2a. While the coordinates θ and φ are defined identically in the two coordinate systems, the physical and mathematical meaning of η which ranges from 0 to ∞ can be obtained from the following relations to the Cartesian coordinates, $\vec{r}(\mathrm{x}, \mathrm{y}, \mathrm{z})$:



$$\mathrm{x} = \mathrm{a} \cdot \sinh\eta \sin\theta \cos\varphi\,; \mathrm{y} = \mathrm{a} \cdot \sinh\eta \sin\theta \sin\varphi\,; \mathrm{z} = \mathrm{a} \cdot \cosh\eta \cos\theta$$

*Note 1: $\sinh\eta = \frac{e^{\eta}-e^{-\eta}}{2}, \cosh\eta = \frac{e^{\eta}+e^{-\eta}}{2}$

i) For each coordinate (r_i; $r_1 = \eta$, $r_2 = \theta$, $r_3 = \varphi$), $\frac{\partial \vec{\mathrm{r}}}{\partial \mathrm{r_i}}$ represents a vector tangent to the 'r_i-curve' which is the intersection of a '$r_{j(\neq i)}$ = constant' surface and a '$r_{k(\neq i \text{ and } \neq j)}$ = constant' surface. Show that the three tangent vectors are orthogonal to each other, which indicates that the prolate spherical coordinates are an orthogonal coordinate system. [10 pts]

ii) The spheroid defined by a '$\eta = \eta_0$' surface can be represented by an equation, $\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Determine the volume of the spheroid in terms of a, b or c, using the prolate coordinate system. [15 pts]

*Note 2: The infinitesimal volume element is given as
dxdydz = |J|dηdθdφ, where |J| is a (Jacobian) determinant defined below:

$$|\mathrm{J}| = \begin{vmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

*Note 3: $\int \sinh^n ax\, dx = \frac{1}{an}\sinh^{n-1}ax \cosh ax - \frac{n-1}{n}\int \sinh^{n-2}ax\, dx;\;$ for n > 0)