Mathematics for Chemistry
2015. 04. 21

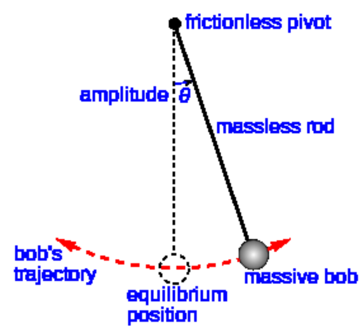
Mid-Term Exam

1. Show that l'Hopital's rule amounts to forming a Maclaurin expansion of both the numerator and the denominator by evaluating the following limit both ways: [10 pts] $\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}$

2. Derive the Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$. [10 pts]

3. The beta function is given as $B(x,y) = \int_0^1 z^{x-1}(1-z)^{y-1}dz$, where both x and y are positive. i) Transform the beta function into the following form by a simple change of variable: $B(x,y) = 2\int_0^{\pi/2}\sin\theta^{2x-1}(\cos\theta)^{2y-1}d\theta$. ii) Derive the relation: $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ for positive integers, m and n. [Hint 1: Start with letting $z = x^2$ for $\Gamma(l) = \int_0^\infty z^{l-1}e^{-z}dz$, form a double integral for $\Gamma(m)\Gamma(n)$ and transform it into the plane polar coordinates. Hint 2: $dxdy = rdrd\theta$. Hint 3: Make use of the results given in (i).] [20 pts]

4. The pendulum to your right oscillates in a viscous medium with an angular amplitude $\boldsymbol{\theta}$ in a single plane. The pendulum support of length $\boldsymbol{l}$ is rigid and massless, and supports a bob of mass $\boldsymbol{m}$. When the viscous force is proportional to the velocity, $\frac{ds}{dt}$ where s is the arc length, the equation of motion can be given as $\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega_0^2\theta = 0$, where γ and ω_0 are the frictional coefficient and the angular frequency when $\gamma = 0$, respectively. Provide the full solutions to the equation of motion under the initial conditions, $\theta(0) = \theta_0$ and $\left[\frac{d\theta}{dt}\right]_{t=0} = 0$. You also need to consider the case where $\gamma^2 = 4\omega_0^2$. [20 pts]



5. Find the power series solution for the differential equation: $[(1-x^2)y'(x)]' + \alpha(\alpha+1)y(x) = 0$, where α is constant. [20 pts]

6. i) Find the Fourier transforms, $\hat{F}(\omega)$ & $\hat{G}(\omega)$, of the following two functions: $f(t) = e^{-\alpha t}$, $g(t) = e^{-\alpha t}\cos\omega_0 t$ (both defined as such for $t \geq 0$, both defined as zero for $t < 0$; α & ω_0 are positive constants), and specify their real and imaginary parts, respectively. ii) Sketch $\mathrm{Re}[\hat{F}(\omega)]$, $\mathrm{Im}[\hat{F}(\omega)]$, $\mathrm{Re}[\hat{G}(\omega)]$ and $\mathrm{Im}[\hat{G}(\omega)]$. [Hint: $\hat{F}(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du$] [20 pts]

7. The Legendre polynomials, $P_n(x)$, and their recursion relations are very useful in spectroscopy. [20 pts]

   i) Derive the following recursion formula using the generating function, $G(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)^{-1/2}$:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \qquad : n \geq 1$$

   ii) Using $\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{(2n+1)}\delta_{nm}$, show the following:

$$\int_{-1}^{1} xP_n(x)P_m(x)dx = \frac{2(n+1)}{(2n+1)(2n+3)}\delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)}\delta_{m,n-1}$$